Exercise 70

Find the value of the number a such that the families of curves $y = (x+c)^{-1}$ and $y = a(x+k)^{1/3}$ are orthogonal trajectories.

Solution

The points of intersection are found by solving the system of equations for x and y.

$$\begin{cases} y = (x+c)^{-1} \\ \\ y = a(x+k)^{1/3} \end{cases}$$

Subtract the respective sides of these equations.

$$0 = (x+c)^{-1} - a(x+k)^{1/3}$$

$$-(x+c)^{-1} = -a(x+k)^{1/3}$$

Square both sides.

$$(x+c)^{-2} = a^2(x+k)^{2/3}$$

Differentiate both sides of the given equations with respect to x.

$$\frac{d}{dx}(y) = \frac{d}{dx}[(x+c)^{-1}] \qquad\qquad \frac{d}{dx}(y) = \frac{d}{dx}[a(x+k)^{1/3}]$$

$$\frac{dy}{dx} = -(x+c)^{-2} \qquad\qquad \frac{dy}{dx} = \frac{a}{3}(x+k)^{-2/3}$$

At any point of intersection $(x+c)^{-2} = a^2(x+k)^{2/3}$, so the slopes of the tangent lines are as follows.

$$\frac{dy}{dx} = -a^2(x+k)^{2/3} \qquad\qquad \frac{dy}{dx} = \frac{a}{3(x+k)^{2/3}}$$

In order for the families of curves, $y = (x+c)^{-1}$ and $y = a(x+k)^{1/3}$, to be orthogonal trajectories, the slopes have to be negative reciprocals.

$$\frac{1}{a^2(x+k)^{2/3}} = \frac{a}{3(x+k)^{2/3}}$$

$$\frac{1}{a^2} = \frac{a}{3}$$

$$3 = a^3$$

$$a = \sqrt[3]{3}$$

Observe that when $a = \sqrt[3]{3}$ the tangent lines at all points of intersection are orthogonal.